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The embedding e_0 and the spectrum-dependent R-matrix for $q-F_4$

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Abstract. The explicit embedding forms of e_0 for the minimal representations of the quantum F_4, E_6 and E_7 are obtained in this paper. In terms of Jimbo's method, the spectrum-dependent R-matrices, both trigonometric and rational, for those representations are computed.

1. Introduction

Finding solutions to the Yang–Baxter equation plays the central role in constructing statistical vertex models and face models [1-3], computing representations of the braid group and link polynomials [4], and evaluating the correlation functions in conformal field theory [5]. Jimbo [6] presented a principle method for computing the spectrum-dependent solutions $R_q(x)$ to the Yang–Baxter equation. However, he gave the embedding forms e_0 only for the quantum $A_n^{(1)}$ [7]. Lack of the explicit expression of e_0 is the last obstacle to the computation for the solutions.

Fortunately, for a given representation of a given quantum Lie universal enveloping algebra, it may be possible to obtain the explicit expression of e_0, especially for minimal representations. Kuniba [8] found this expression for e_0 for the minimal (seven-dimensional) representation of the quantum G_2. We [9, 10] computed the solutions of the Yang–Baxter equation without a spectral parameter for the minimal representations of the quantum E_6, E_7 and F_4. Owing to lack of the expression of e_0, we [11-13] computed the spectrum-dependent solutions for the minimal representations of the quantum E_6 and E_7 in terms of the limit conditions. As a reasonable extension, in the present paper we are going to find out the embedding expression e_0 for the minimal representations of the quantum F_4 as well as E_6 and E_7, and then to compute the spectrum-dependent solutions $R_q(x)$, both trigonometric and rational, for those representations. Of course, the solutions for the quantum E_6 and E_7 coincide with those given in [11]. The explicit forms of the quantum projectors for the quantum E_6, E_7 and F_4 will be published elsewhere [9, 10]. The spectrum-dependent solutions for the minimal representations of some quantum Lie enveloping algebras were listed in [14], but with a misprint for the quantum F_4. The rational solutions for the non-exceptional Lie algebras were discussed in [15].

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The organization of this paper is as follows. In section 2, we review Jimbo's method. The explicit embedding expression e_0 for the minimal representation of the quantum F_4 is given in section 3. The spectrum-dependent trigonometric and rational solutions to the Yang–Baxter equation for that representation are computed in sections 3 and 4, respectively. In sections 5 and 6 we obtain the explicit expressions e_0 for the minimal representations of the quantum E_6 and E_7, and show that the solutions given in the previous paper [11] are correct.

2. Jimbo's method

For a simple Lie algebra $\mathscr{L}$ with rank l, there are l simple roots r_j and l fundamental weights $\lambda_j, j=1,2,\ldots,l$. The irreducible representation is denoted by its highest weight N, and the states by the weight m. Both N and m are expressed in the integral combinations of λ_j:

$$N=\sum_{j=1}^{l}(N)_j\lambda_j \qquad m=\sum_{j=1}^{l}(m)_j\lambda_j. \tag{1}$$

Let r_0 be the lowest negative root

$$r_0=\sum_{j=1}^{l}\alpha_j r_j. \tag{2}$$

The Cartan matrix is now defined:

$$a_{ij}=\frac{2(r_i,r_j)}{(r_i,r_i)} \qquad i,j=0,1,2,\ldots,l \tag{3}$$

where (r_i, r_j) denotes the inner product. In this paper we take the convention that the length of the shorter root is normalized to be unity.

For quantization, a quantum parameter q, which is not a root of unity, is introduced. Then $q_j=q^{(r_j,r_j)/2}$ is defined. Now, for a given irreducible representation N_0 of the quantum Lie universal enveloping algebra $q-\mathscr{L}$, there are generators $e_j\equiv D_q^{N_0}(e_j)$, $f_j\equiv D_q^{N_0}(f_j)$ and $h_j\equiv D_q^{N_0}(h_j)$ (or $k_j\equiv D_q^{N_0}(k_j)=q_j^{h_j}$ instead), where $j=1,2,\ldots,l$, which satisfy the standard quantum algebraic relations [16]

$$\begin{aligned}
&[h_i,h_j]=0\\
&[h_i,e_j]=a_{ij}e_j \qquad [h_i,f_j]=-a_{ij}f_j\\
&[e_i,f_j]=\delta_{ij}\frac{q_i^{2h_i}-q_i^{-2h_i}}{q_i^2-q_i^{-2}}=\delta_{ij}\frac{k_i^2-k_i^{-2}}{q_i^2-q_i^{-2}}\\
&\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_{q_i^2} e_i^{1-a_{ij}-n}e_je_i^n=0 \qquad i\neq j\\
&\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_{q_i^2} f_i^{1-a_{ij}-n}f_jf_i^n=0 \qquad i\neq j
\end{aligned} \tag{4}$$

where $i,j=1,2,\ldots,l$, and

$$\begin{aligned}
&[m]_q=\frac{q^m-q^{-m}}{q-q^{-1}}\\
&\begin{bmatrix}n\\ m\end{bmatrix}_q=\frac{[n]_q[n-1]_q\ldots[n-m+1]_q}{[m]_q[m-1]_q\ldots[1]_q}.
\end{aligned} \tag{5}$$

The co-product in $V_{N_0}\otimes V_{N_0}$ defined as

$$\begin{aligned}\Delta(h_j)&=h_j\otimes\mathbb{1}+\mathbb{1}\otimes h_j\\ \Delta(k_j)&=k_j\otimes k_j\\ \Delta(e_j)&=e_j\otimes k_j^{-1}+k_j\otimes e_j\\ \Delta(f_j)&=f_j\otimes k_j^{-1}+k_j\otimes f_j\end{aligned}\tag{6}$$

is a representation of $q-\mathscr{L}$, but generally not irreducible. It can be reduced by an orthogonal quantum Clebsch–Gordan matrix C_q,

$$C_q^{-1}\Delta(I_\alpha)C_q=\bigoplus_\mu D_q^{N_\mu}(I_\alpha)\qquad I_\alpha=e_j,f_j\text{ or }h_j\tag{7}$$

$(C_q)_{n_1n_2Nm}$ is a $d_{N_0}^2\times d_{N_0}^2$ orthogonal matrix with the row indices n_1n_2 and column indices Nm, where d_{N_0} is the dimension of N_0, and N is one of N_μ in the Clebsch–Gordan series. The submatrix $(C_q)_N$, which reduces the co-product into the irreducible representation N, is a $d_{N_0}^2\times d_N$ matrix with the row indices n_1n_2 and column index m. Generally, the decomposition of the co-product is not multiplicity free, i.e. some N_μ may be equal to each other.

Then

$$\begin{aligned}h_0&=\sum_{j=1}^{l}\frac{(r_j,r_j)}{(r_0,r_0)}\alpha_jh_j\equiv D_q^{N_0}(h_0)\\ k_0&=q^{h_0(r_0,r_0)/2}\equiv D_q^{N_0}(k_0)\end{aligned}\tag{8}$$

is defined.

It is assumed that the generators $e_0\equiv D_q^{N_0}(e_0)$ and $f_0\equiv D_q^{N_0}(f_0)$ exist in the representation space V_{N_0} so that the quantum algebraic relations (4) are satisfied for $i,j=0,1,2,\ldots,l$. The generators e_0 and f_0 exist, at least for some irreducible representations of some quantum Lie enveloping algebras. For example, Jimbo [7] gave the explicit form e_0 for $q-A_n$, Kuniba [8] gave e_0 for the minimal representation of $q-G_2$. In this paper the generators for the minimal representations of $q-F_4$, $q-E_6$ and $q-E_7$ are given explicitly. These algebras may be called the embedding of the deformation of the corresponding Kac–Moody enveloping algebras. However, there are no definitions for the co-product of e_0 and f_0, and no central extension.

For the linear system

$$[R_q(x),h_j\otimes\mathbb{1}+\mathbb{1}\otimes h_j]=0\tag{9a}$$

$$(e_j\otimes k_j+k_j^{-1}\otimes e_j)R_q(x)=R_q(x)(e_j\otimes k_j^{-1}+k_j\otimes e_j)\tag{9b}$$

$$(xe_0\otimes k_0+k_0^{-1}\otimes e_0)R_q(x)=R_q(x)(xe_0\otimes k_0^{-1}+k_0\otimes e_0)$$

$$(f_j\otimes k_j+k_j^{-1}\otimes f_j)R_q(x)=R_q(x)(f_j\otimes k_j^{-1}+k_j\otimes f_j)\tag{9c}$$

$$(x^{-1}f_0\otimes k_0+k_0^{-1}\otimes f_0)R_q=R_q(x)(x^{-1}f_0\otimes k_0^{-1}+k_0\otimes f_0)$$

Jimbo proved [6] the following.

(i) For a general value of q, the dimension of the solution space of the linear system (9) is at most one.

(ii) A solution of (9*b*) satisfies (9*a*) and (9*c*).

(iii) A solution of (9) satisfies the Yang–Baxter equation

$$R_q^{12}(x)R_q^{13}(xy)R_q^{23}(y)=R_q^{23}(y)R_q^{13}(xy)R_q^{12}(x).\tag{10}$$

Let

$$\check{R}_q(x)=PR_q(x)\tag{11}$$

where P denotes the transposition, P: $V_1 \otimes V_2 \to V_2 \otimes V_1$, $P(u \otimes v) = v \otimes u$. Now (9$b$) becomes

$$[\check{R}_q(x), \Delta(I_\alpha)] = 0 \tag{12a}$$

$$(xk_0 \otimes e_0 + e_0 \otimes k_0^{-1})\check{R}_q(x) = \check{R}_q(x)(k_0 \otimes e_0 + xe_0 \otimes k_0^{-1}) \tag{12b}$$

where I_α denotes e_j, f_j or h_j, $j = 1, 2, \ldots, l$.

Owing to the Schur theorem, from (12a) we have

$$\check{R}_q(x)(C_q)_{N\mu} = \sum_\nu (C_q)_{N_\nu} \Lambda_{N_\nu N_\mu}(x, q) \tag{13}$$

where the summed irreducible representation N_ν is equal to N_μ. If N_μ appears in the Clebsch–Gordan series only once, then only one term, $\nu = \mu$, appears in the RHS of (13). Because of orthogonality of the quantum Clebsch–Gordan matrix, we obtain the expression for the spectrum-dependent solution $\check{R}_q(x)$ to the Yang–Baxter equation:

$$\check{R}_q(x) = \sum_{\nu\mu} (C_q)_{N_\nu} \Lambda_{N_\nu N_\mu}(x, q)(\tilde{C}_q)_{N_\mu}$$
$$N_\nu = N_\mu \tag{14}$$

Now, the primary object is to find $\Lambda_{N_\nu N_\mu}$.

Substituting (14) into (12b), we have

$$\sum_{N''=N} (xX(q)_{N'm',N''m} + Y(q)_{N'm',N''m})\Lambda_{N''N}(x, q)$$
$$= \sum_{N''=N'} \Lambda_{N'N''}(x, q)(X(q)_{N''m',Nm} + xY(q)_{N''m',Nm}) \tag{15}$$

where

$$X(q)_{N'm',Nm} = \sum_{\substack{n_1 n_2 \\ n_1' n_2'}} (C_q)_{n_1' n_2' N'm'} D_q^{N_0}(k_0)_{n_1' n_1} D_q^{N_0}(e_0)_{n_2' n_2} (C_q)_{n_1 n_2 Nm}$$
$$Y(q)_{N'm',Nm} = \sum_{\substack{n_1 n_2 \\ n_1' n_2'}} (C_q)_{n_1' n_2' N'm'} D_q^{N_0}(e_0)_{n_1' n_1} D_q^{N_0}(k_0^{-1})_{n_2' n_2} (C_q)_{n_1 n_2 Nm}. \tag{16}$$

We use $D_q^{N_0}(e_0)$ and $D_q^{N_0}(k_0)$ to emphasize the representation matrices of e_0 and k_0 in N_0. Since e_0 corresponds to the lowest negative root r_0, in (16) we have

$$m' = m + r_0. \tag{17}$$

$\Lambda_{N'N}(x, q)$ is independent of m and m'. Equation (15) is overdetermined for $\Lambda_{N'N}(x, q)$. The existence of $\check{R}_q(x)$ means that equation (15) should be consistent. Because $[e_0, f_j] = 0$, $j \neq 0$, it has been proved [11] that both $k_0 \otimes e_0$ and $e_0 \otimes k_0^{-1}$ are commutable with $\Delta(f_j)$, $j \neq 0$. Therefore, if a solution $\Lambda_{N'N}(x, q)$ satisfies (15) with $m = N$ (the highest weight), this solution must satisfy (15) with any m.

If the explicit form e_0 in the representation N_0 is known, the spectrum-dependent solution $\check{R}_q(x)$ to the Yang–Baxter equation for the representation N_0 can be computed by solving (15) with $m = N$. There is no principle obstacle for the cases where the decomposition of the co-product is not multiplicity free. A typical example with multiplicity is the octet representation of $q - \mathrm{sl}(3)$. The detailed computation of $\check{R}_q(x)$ for the octet representation was given in our previous paper [17]. In the present paper, we compute $\check{R}_q(x)$ for the minimal representations of $q - F_4$, $q - E_6$ and $q - E_7$. In these cases the decompositions are multiplicity free so that only one term appears in the summations of (13), (14) and (15), respectively, and $\Lambda(x, q)$ is diagonal, $\Lambda_{N'N}(x, q) = \delta_{N'N}\Lambda_N(x, q)$.

3. Trigonometric solution for $q-F_4$

First of all we list some relevant properties of the Lie algebra F_4. The Dynkin diagram of F_4 is given in figure 1. The Cartan matrix is as follows:

$$a=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad a^{-1}=\begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{pmatrix}. \tag{18}$$

The decomposition of the direct product of the minimal representation $N_0=\lambda_4=(0001)$ is multiplicity free:

$$(0001)\otimes(0001)=(0002)\oplus(0010)\oplus(1000)\oplus(0001)\oplus(0000) \tag{19a}$$

or briefly

$$N_0\otimes N_0=N_1\oplus N_2\oplus N_3\oplus N_4\oplus N_5 \tag{19b}$$

where $N_1=2\lambda_4$, $N_2=\lambda_3$, $N_3=\lambda_1$, $N_4=N_0$ and $N_5=0$. The Casimir operator $C_2(N)$ can be calculated by the inner product $(N, N+2\rho)$ where $\rho=\Sigma_j\lambda_j$:

$$\begin{aligned} &C_2(N_0)=C_2(N_4)=12 \qquad C_2(N_1)=26 \\ &C_2(N_2)=24 \qquad C_2(N_3)=18 \qquad C_2(N_5)=0. \end{aligned} \tag{20}$$

Since N_3 is the adjoint representation, the lowest negative root r_0 is

$$r_0=-\lambda_1=-2r_1-3r_2-4r_3-2r_4. \tag{21}$$

In order to simplify the notation, we enumerate the states in N_0 as shown in table 1.

For the quantum F_4 universal enveloping algebra, the representation matrices of e_j, f_j and h_j in N_0 are given in the following:

$$\begin{aligned}
e_1=\tilde{f}_1&=E_{4\,5}+E_{6\,7}+E_{8\,10}+E_{17\,19}+E_{20\,21}+E_{22\,23}\\
e_2=\tilde{f}_2&=E_{3\,4}+E_{7\,9}+E_{10\,12}+E_{15\,17}+E_{18\,20}+E_{23\,24}\\
e_3=\tilde{f}_3&=E_{2\,3}+E_{4\,6}+E_{5\,7}+E_{9\,11}+[2]^{1/2}E_{12\,14}+[2]^{1/2}E_{14\,15}+E_{16\,18}\\
&\quad+E_{20\,22}+E_{21\,23}+E_{24\,25}\\
e_4=\tilde{f}_4&=E_{1\,2}+E_{6\,8}+E_{7\,10}+E_{9\,12}+[2]^{-1/2}E_{11\,14}+([3]/[2])^{1/2}E_{11\,13}+([3]/[2])^{1/2}E_{13\,16}\\
&\quad+[2]^{-1/2}E_{14\,16}+E_{15\,18}+E_{17\,20}+E_{19\,21}+E_{25\,26}\\
h_1&=E_{4\,4}-E_{5\,5}+E_{6\,6}-E_{7\,7}+E_{8\,8}-E_{10\,10}+E_{17\,17}-E_{19\,19}+E_{20\,20}-E_{21\,21}+E_{22\,22}-E_{23\,23}\\
h_2&=E_{3\,3}-E_{4\,4}+E_{7\,7}-E_{9\,9}+E_{10\,10}-E_{12\,12}+E_{15\,15}-E_{17\,17}+E_{18\,18}\\
&\quad-E_{20\,20}+E_{23\,23}-E_{24\,24}\\
h_3&=E_{2\,2}-E_{3\,3}+E_{4\,4}+E_{5\,5}-E_{6\,6}-E_{7\,7}+E_{9\,9}-E_{11\,11}+2E_{12\,12}-2E_{15\,15}+E_{16\,16}\\
&\quad-E_{18\,18}+E_{20\,20}+E_{21\,21}-E_{22\,22}-E_{23\,23}+E_{24\,24}-E_{25\,25}\\
h_4&=E_{1\,1}-E_{2\,2}+E_{6\,6}+E_{7\,7}-E_{8\,8}+E_{9\,9}-E_{10\,10}+2E_{11\,11}-E_{12\,12}+E_{15\,15}\\
&\quad-2E_{16\,16}+E_{17\,17}-E_{18\,18}+E_{19\,19}-E_{20\,20}-E_{21\,21}+E_{25\,25}-E_{26\,26}
\end{aligned} \tag{22}$$

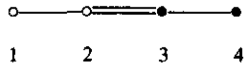


Figure 1. Dynkin diagram of F_4.

where the tilde denotes transpose,

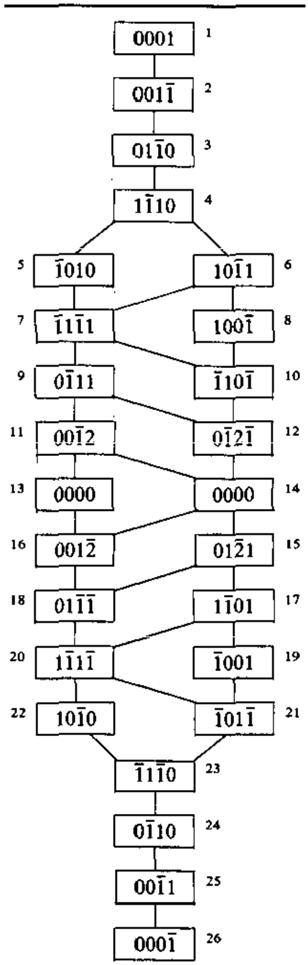
$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl} \tag{23}$$

and hereafter the square bracket $[m]$ without a subscript is used to denote that with a subscript q:

$$[m] \equiv [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{24}$$

In calculating these matrices we use the properties of the subalgebras $q-\mathrm{sl}(2)$ in $q-F_4$. The representation matrices of the generators of $q-\mathrm{sl}(2)$ are well known. For

Table 1. Enumeration of 26 states in the minimal representation N_0 of F_4.



the representation with the highest weight N of $q-\mathrm{sl}(2)$, we have

$$h|m\rangle = m|m\rangle$$

$$e|m-2\rangle = \Gamma_m^N(q)|m\rangle \qquad f|m\rangle = \Gamma_m^N(q)|m-2\rangle$$

$$\Gamma_m^N(q) = \left[\frac{N+m}{2}\right]\left[\frac{N-m}{2}+1\right].$$

There are four independent subalgebras $q-\mathrm{sl}(2)$. If the state in the representation N_0 of $q-F_4$ is a simple weight, the matrix elements of the generators of $q-F_4$ are determined completely by the properties of the subalgebras. When the states are multiple, those states should be combined so that the states belong to the multiplets of the subalgebras. In the representation N_0, the states 13 and 14 are multiple (see table 1). We choose the state 13 to be a singlet of the third subalgebra $q-\mathrm{sl}(2)_3$ spanned by e_3, f_3 and h_3, and the state 14 to belong to a triplet with the states 12 and 15:

$$\begin{aligned} &f_3|12\rangle = [2]^{1/2}|14\rangle \qquad f_3|14\rangle = [2]^{1/2}|15\rangle \qquad f_3|13\rangle = f_3|15\rangle = 0 \\ &e_3|15\rangle = [2]^{1/2}|14\rangle \qquad e_3|14\rangle = [2]^{1/2}|12\rangle \qquad e_3|12\rangle = e_3|13\rangle = 0. \end{aligned} \tag{25}$$

For the fourth subalgebra $q-\mathrm{sl}(2)_4$ spanned by e_4, f_4 and h_4 both states 13 and 14 are the combinations of the triplet and singlet. Suppose that

$$f_4|11\rangle = a|13\rangle + b|14\rangle.$$

From (25) and (1) we have

$$f_3e_3f_4|11\rangle = [2]b|14\rangle = f_3f_4e_3|11\rangle = f_3f_4|9\rangle = f_3|12\rangle = [2]^{1/2}|14\rangle.$$

Therefore, $b=[2]^{-1/2}$. From $a^2+b^2=[2]$ we have $a=([3]/[2])^{1/2}$. The method given above can be generalized to determine the representation matrix elements of any quantum Lie enveloping algebra.

From (8) and (21) we have

$$\begin{aligned} h_0 = &-2h_1-3h_2-2h_3-h_4 \\ &= -E_{1\,1}-E_{2\,2}-E_{3\,3}-E_{4\,4}-E_{6\,6}-E_{8\,8}+E_{19\,19}+E_{21\,21}+E_{23\,23} \\ &\quad +E_{24\,24}+E_{25\,25}+E_{26\,26} \end{aligned} \tag{26}$$

$$k_0 = q^{h_0}.$$

From (21) e_0 can relate those states m and m' satisfying $m'=m+r_0$. When $q=1$ (Lie algebra F_4) we have

$$e_0 = E_{19\,1}+E_{21\,2}+E_{23\,3}+E_{24\,4}+E_{25\,6}+E_{26\,8}. \tag{27}$$

It is easy to check that, even $q\neq 1$, (22), (26) and (27) satisfy the quantum algebraic relation (4) with $j=0,1,2,3,4$. In other words, the embedding e_0 for $q\neq 1$ is the same as that for $q=1$. This property also holds for the minimal representations of the quantum A_n, B_n (spinor), C_n, D_n (vector and two spinor), G_2, F_4, E_6 and E_7 universal enveloping algebras, even though the eigenvalues of h_j may not be equal to ± 1 or 0.

Now, in terms of the explicit forms (26) and (27) for e_0 and k_0 and the quantum Clebsch-Gordan coefficients computed in [10], we can compute $X(q)_{N'(N+r_0),NN}$ and $Y(q)_{N'(N+r_0),NN}$ in (16). Note that the weight $(N+r_0)$ in the irreducible representation

N' may be multiple, so we have to compute for all weights $(N+r_0)$ one by one. Through a tedious calculation we obtain the non-vanishing elements of $X(q)$ and $Y(q)$ satisfying the following relations:

$$X(q)_{N(m+r_0),Nm} = Y(q)_{N(m+r_0),Nm} \tag{28a}$$

where $N = N_1, N_2, N_3, N_4$ and N_5, respectively, and

$$X(q)_{N'(m+r_0),Nm} = -q^{C_2(N')-C_2(N)}Y(q)_{N'(m+r_0),Nm} \tag{28b}$$

where the pair (N, N') or (N', N) denotes (N_1, N_2), (N_1, N_3), (N_2, N_4) or (N_3, N_5). Therefore, from (15) we have

$$\begin{aligned}
(x-q^2)\Lambda_{N_1}(x,q) &= (1-xq^2)\Lambda_{N_2}(x,q)\\
(x-q^8)\Lambda_{N_1}(x,q) &= (1-xq^8)\Lambda_{N_3}(x,q)\\
(x-q^{12})\Lambda_{N_2}(x,q) &= (1-xq^{12})\Lambda_{N_4}(x,q)\\
(x-q^{18})\Lambda_{N_3}(x,q) &= (1-xq^{18})\Lambda_{N_5}(x,q).
\end{aligned} \tag{29}$$

Choosing the arbitrary factor in $\check{R}_q(x)$ so that

$$\Lambda_{N_1}(x,q) = (1-xq^2)(1-xq^8)(1-xq^{12})(1-xq^{18}) \tag{30}$$

we have the spectrum-dependent solution $\check{R}_q(x)$ to the Yang–Baxter equation for the minimal representation (0001) of $q-F_4$ as follows:

$$\begin{aligned}
\check{R}_q(x) = {} & (1-xq^2)(1-xq^8)(1-xq^{12})(1-xq^{18})(C_q)_{N_1}(\tilde{C}_q)_{N_1}\\
& +(x-q^2)(1-xq^8)(1-xq^{12})(1-xq^{18})(C_q)_{N_2}(\tilde{C}_q)_{N_2}\\
& +(1-xq^2)(x-q^8)(1-xq^{12})(1-xq^{18})(C_q)_{N_3}(\tilde{C}_q)_{N_3}\\
& +(x-q^2)(1-xq^8)(x-q^{12})(1-xq^{18})(C_q)_{N_4}(\tilde{C}_q)_{N_4}\\
& +(1-xq^2)(x-q^8)(1-xq^{12})(x-q^{18})(C_q)_{N_5}(\tilde{C}_q)_{N_5}.
\end{aligned} \tag{31}$$

Obviously, we have

$$\check{R}_q(0) = \check{R}_q = \sum_{\mu} \xi_\mu q^{C_2(N_1)-C_2(N_\mu)}(C_q)_{N_\mu}(\tilde{C}_q)_{N_\mu} \tag{32a}$$

$$x^{-4}\check{R}_q(x)|_{x=\infty} = q^{40}\check{R}_q^{-1} = \sum_{\mu} \xi_\mu q^{40-C_2(N_1)+C_2(N_\mu)}(C_q)_{N_\mu}(\tilde{C}_q)_{N_\mu} \tag{32b}$$

where $\xi_1 = -\xi_2 = -\xi_3 = \xi_4 = \xi_5 = 1$ and $C_2(N_\mu)$ was given in (20). We also have

$$\begin{aligned}
&\check{R}_q(x)\check{R}_q(x^{-1})\\
&\quad = x^{-4}(1-xq^2)(x-q^2)(1-xq^8)(x-q^8)(1-xq^{12})\\
&\qquad \times(x-q^{12})(1-xq^{18})(x-q^{18})\mathbb{1}.
\end{aligned} \tag{33}$$

4. Rational solution for $q-F_4$

A rational solution to the Yang–Baxter equation can be obtained from a spectrum-dependent trigonometric one through an appropriate limit process [16, 18]. Letting

$x=q^{2u/\eta}$ and taking the limit $q\to 1$, we have

$$\begin{aligned}
R(u,\eta)=P\check{R}(u,\eta)&=\lim_{q\to 1}P\check{R}_q(q^{2u/\eta})/(1-q^{2u/\eta})^4\\
&=P\{(1+20\eta/u+133\eta^2/u^2+330\eta^3/u^3+216\eta^4/u^4)P_1\\
&\quad+(-1-18\eta/u-95\eta^2/u^2-102\eta^3/u^3+216\eta^4/u^4)P_2\\
&\quad+(-1-12\eta/u-5\eta^2/u^2+222\eta^3/u^3+216\eta^4/u^4)P_3\\
&\quad+(1+6\eta/u-49\eta^2/u^2-174\eta^3/u^3+216\eta^4/u^4)P_4\\
&\quad+(1-6\eta/u-49\eta^2/u^2+174\eta^3/u^3+216\eta^4/u^4)P_5\}\\
&=\mathbb{1}+(2t+18\,\mathbb{1})\eta/u+(99\,\mathbb{1}+4P+30t+28P_4+208P_5)\eta^2/u^2\\
&\quad+(162\,\mathbb{1}+60P+108t+252P_4+1248P_5)\eta^3/u^3+216P\eta^4/u^4
\end{aligned}\tag{34}$$

where

$$P_\mu=(C_q)_{N_\mu}(\tilde{C}_q)_{N_\mu}|_{q=1}=C_{N_\mu}\tilde{C}_{N_\mu}$$

and C_μ is the usual CG coefficients of F_4.

$$\begin{aligned}
&\mathbb{1}=\sum_{\mu=1}^{j}P_\mu \qquad P=P_1-P_2-P_3+P_4+P_5\\
&2t=\sum_a I_a\otimes I_a \qquad C_2(N_0)\mathbb{1}=\sum_a I_a^2\\
&2tP_\mu=\{C_2(N_\mu)-2C_2(N_0)\}P_\mu.
\end{aligned}\tag{35}$$

I_a is the orthogonal basis of F_4 in the representation N_0. In terms of the explicit Clebsch–Gordan coefficients for F_4 we have

$$(P_5)_{m_1m_2,m_1'm_2'}=(26)^{-1}(-1)^{\kappa(m_1)+\kappa(m_1')}\delta_{m_1\bar{m}_2}\delta_{m_1'\bar{m}_2'}\tag{36}$$

where $\kappa(m_1)=\Sigma_j p_j$ if $N_0-m_1=\Sigma_j p_j r_j$, and the enumeration of m and $\bar{m}$ satisfy

$$\bar{m}=\begin{cases} m & \text{when } m=13 \text{ or } 14\\ 27-m & \text{when } m\neq 13 \text{ or } 14.\end{cases}\tag{37}$$

Hereafter we use the same m to denote the weight or enumeration of the state.

The form of P_4 is more complicated than P_5,

$$\begin{aligned}
(P_4)_{m_1m_2,m_1'm_2'}&=\sum_m C_{m_1m_2N_4m}C_{m_1'm_2'N_4m}\\
&=(P_4)_{m_1'm_2',m_1m_2}=(P_4)_{m_2m_1,m_1'm_2'}\\
&=(P_4)_{\bar{m}_1\bar{m}_2,\bar{m}_1'\bar{m}_2'}
\end{aligned}\tag{38}$$

where the summation occurs only for the multiple weights, i.e. for $m=13$ and 14. P_4 contains one 28×28 submatrix and twenty-four 12×12 submatrices, and the rest of the components of P_4 are vanishing. There are three different patterns of the 12×12 submatrices, and each pattern is shared by eight 12×12 submatrices.

(i) 28×28 submatrix. The indices of rows and columns are $(m, \bar{m})$ and (13, 14), (14, 13). Because of the symmetry (38), it is only needed to list the following independent components:

$$(P_4)_{m_1\bar{m}_1,13\,14}=(P)_{n_1\bar{n}_1,13\,14}=0$$

$$(P_4)_{m_2\bar{m}_2,13\,14}=(P_4)_{m_3\bar{m}_3,13\,14}=\tfrac{\sqrt{3}}{28}$$

$$(P_4)_{n_2\bar{n}_2,13\,14}=(P_4)_{n_3\bar{n}_3,13\,14}=-\tfrac{\sqrt{3}}{28}$$

$$(P_4)_{m_j\bar{m}_j,m'_j\bar{m}'_j}=(P_4)_{n_j\bar{n}_j,n'_j\bar{n}'_j}$$
$$=-(P_4)_{m_j\bar{m}_j,n_j\bar{n}_j}=\tfrac{1}{14}\qquad j=1,2,3$$

$$(P_4)_{m_1\bar{m}_1,m_2\bar{m}_2}=(P_4)_{m_1\bar{m}_1,n_3\bar{n}_3}=(P_4)_{n_1\bar{n}_1,n_2\bar{n}_2}$$
$$=(P_4)_{n_1\bar{n}_1,m_3\bar{m}_3}=(P_4)_{m_2\bar{m}_2,m_3\bar{m}_3}$$
$$=(P_4)_{n_2\bar{n}_2,n_3\bar{n}_3}=\tfrac{1}{28}$$

$$(P_4)_{m_1\bar{m}_1,n_2\bar{n}_2}=(P_4)_{m_1\bar{m}_1,m_3\bar{m}_3}=(P_4)_{n_1\bar{n}_1,m_2\bar{m}_2}$$
$$=(P_4)_{n_1\bar{n}_1,n_3\bar{n}_3}=(P_4)_{m_2\bar{m}_2,n_3\bar{n}_3}$$
$$=(P_4)_{n_2\bar{n}_2,m_3\bar{m}_3}=-\tfrac{1}{28}$$

where: $m_1=1, 10, 14$; $n_1=8, 12, 13$; $m_2=2, 7$; $n_2=6, 9$, $m_3=3, 5$ and $n_3=4, 11$.

(ii) The first pattern of 12×12 submatrices. The indices of rows and columns of each submatrices are listed as follows:

$$(1, 13)(1, 14)(2, 11)(3, 9)(4, 7)(5, 6)\ldots$$
$$(8, 13)(8, 14)(6, 16)(4, 18)(3, 20)(2, 22)\ldots$$
$$(10, 13)(10, 14)(7, 16)(5, 18)(3, 21)(2, 23)\ldots$$
$$(12, 13)(12, 14)(9, 16)(5, 20)(4, 21)(2, 24)\ldots$$

and those obtained by replacing m to $\bar{m}$. Hereafter, the dots denote the second half states obtained by replacing (m_1, m_2) to (m_2, m_1). For the first submatrix we have

$$(P_4)_{1\,14,m_1m_2}=0\qquad (P_4)_{1\,13,1\,13}=\tfrac{1}{14}$$

$$-(P_4)_{1\,13,2\,11}=(P_4)_{1\,13,3\,9}=-(P_4)_{1\,13,4\,7}$$
$$=(P_4)_{1\,13,5\,6}=\tfrac{\sqrt{6}}{28}$$

$$(P_4)_{2\,11,2\,11}=(P_4)_{2\,11,4\,7}=(P_4)_{3\,9,3\,9}=(P_4)_{3\,9,5\,6}$$
$$=(P_4)_{4\,7,4\,7}=(P_4)_{5\,6,5\,6}=-(P_4)_{2\,11,3\,9}$$
$$=-(P_4)_{2\,11,5\,6}=-(P_4)_{3\,9,4\,7}=-(P_4)_{4\,7,5\,6}=\tfrac{3}{28}.$$

(iii) The second pattern of 12×12 submatrices. The indices of rows and columns of each submatrix are listed as follows:

$$(2, 13)(2, 14)(1, 16)(3, 12)(4, 10)(5, 8)\ldots$$
$$(6, 13)(6, 14)(4, 15)(8, 11)(3, 17)(1, 22)\ldots$$
$$(7, 13)(7, 14)(5, 15)(10, 11)(3, 19)(1, 23)\ldots$$
$$(9, 13)(9, 14)(5, 17)(12, 11)(4, 19)(1, 24)\ldots$$

and those obtained by replacing m to $\bar{m}$. For the first submatrix we have

$$(P_4)_{2\,13,1\,16}=(P_4)_{2\,13,3\,12}=-(P_4)_{2\,13,4\,10}=(P_4)_{2\,13,5\,8}=-\tfrac{\sqrt{6}}{56}$$
$$(P_4)_{2\,14,1\,16}=(P_4)_{2\,14,3\,12}=-(P_4)_{2\,14,4\,10}=(P_4)_{2\,14,5\,8}=-3\tfrac{\sqrt{2}}{56}$$
$$(P_4)_{2\,13,2\,13}=\tfrac{1}{56}\qquad (P_4)_{2\,14,2\,14}=\tfrac{3}{56}$$
$$(P_4)_{2\,13,2\,14}=\tfrac{\sqrt{3}}{56}$$
$$(P_4)_{1\,16,1\,16}=(P_4)_{1\,16,3\,12}=-(P_4)_{1\,16,4\,10}=(P_4)_{1\,16,5\,8}$$
$$=(P_4)_{3\,12,3\,12}=-(P_4)_{3\,12,4\,10}=(P_4)_{3\,12,5\,8}$$
$$=(P_4)_{4\,10,4\,10}=-(P_4)_{4\,10,5\,8}=(P_4)_{5\,8,5\,8}=\tfrac{3}{28}.$$

(iv) The third pattern of 12×12 submatrices. The indices of rows and columns of each submatrix are listed as follows:

(3, 13)(3, 14)(1, 18)(2, 15)(6, 10)(7, 8) . . .

(4, 13)(4, 14)(1, 20)(2, 17)(6, 12)(9, 8) . . .

(5, 13)(5, 14)(1, 21)(2, 19)(7, 12)(9, 10) . . .

(11, 13)(11, 14)(7, 17)(9, 15)(6, 19)(1, 25) . . .

and those obtained by replacing m to $\bar{m}$. For the first submatrix we have

$$-(P_4)_{3\,13,1\,18}=(P_4)_{3\,13,2\,15}=(P_4)_{3\,13,6\,10}=-(P_4)_{3\,13,7\,8}=\tfrac{\sqrt{6}}{56}$$
$$(P_4)_{3\,14,1\,18}=-(P_4)_{3\,14,2\,15}=-(P_4)_{3\,14,6\,10}=(P_4)_{3\,14,7\,8}=3\tfrac{\sqrt{2}}{56}$$
$$(P_4)_{3\,13,3\,13}=\tfrac{1}{56}\qquad (P_4)_{3\,14,3\,14}=\tfrac{3}{56}$$
$$(P_4)_{3\,13,3\,14}=-\tfrac{\sqrt{3}}{56}$$
$$(P_4)_{1\,18,1\,18}=-(P_4)_{1\,18,2\,15}=-(P_4)_{1\,18,6\,10}=(P_4)_{1\,18,7\,8}$$
$$=(P_4)_{2\,15,2\,15}=(P_4)_{2\,15,6\,10}=-(P_4)_{2\,15,7\,8}$$
$$=(P_4)_{6\,10,6\,10}=-(P_4)_{6\,10,7\,8}=(P_4)_{7\,8,7\,8}=\tfrac{3}{28}.$$

5. Solution for $q-E_6$

The Dynkin diagram of E_6 is shown in figure 2.

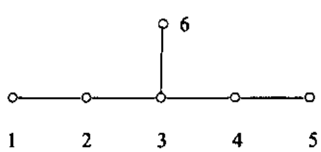


Figure 2. Dynkin diagram of E_6.

The decomposition of the direct product of the minimal representation $N_0=\lambda_1=(100000)$ is multiplicity free:

$$(100000)\otimes(100000)=(200000)\oplus(010000)\oplus(000010) \qquad (39a)$$

or briefly

$$N_0\otimes N_0=N_1\oplus N_2\oplus N_3 \qquad (39b)$$

where $N_1 = 2\lambda_1$, $N_2 = \lambda_2$ and $N_3 = \lambda_5 = N_0^*$. The Casimir operator $C_2(N)$ is given as follows:

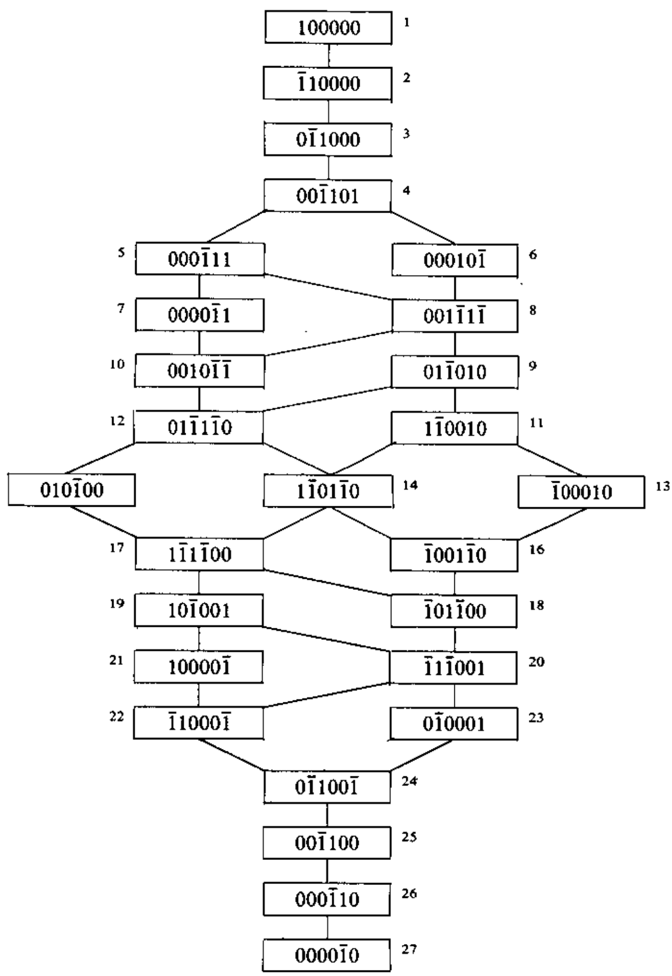
$$C_2(N_0) = C_2(N_3) = \tfrac{26}{3}, \qquad C_2(N_1) = \tfrac{56}{3}, \qquad C_2(N_2) = \tfrac{50}{3}. \tag{40}$$

The lowest negative root r_0 is

$$r_0 = -\lambda_6 = -r_1 - 2r_2 - 3r_3 - 2r_4 - r_5 - 2r_6. \tag{41}$$

In order to simplify the notation, we enumerate the states in N_0 as shown in table 2.

Table 2. Enumeration of 27 states in the minimal representation N_0 of E_6.



For the quantum E_6 universal enveloping algebra, the representation matrices of e_j, f_j and h_j in N_0 are given as follows:

$e_1 = \tilde{f}_1 = E_{1\,2} + E_{11\,13} + E_{14\,16} + E_{17\,18} + E_{19\,20} + E_{21\,22}$

$e_2 = \tilde{f}_2 = E_{2\,3} + E_{9\,11} + E_{12\,14} + E_{15\,17} + E_{20\,23} + E_{22\,24}$

$e_3 = \tilde{f}_3 = E_{3\,4} + E_{8\,9} + E_{10\,12} + E_{17\,19} + E_{18\,20} + E_{24\,25}$

$e_4 = \tilde{f}_4 = E_{4\,5} + E_{6\,8} + E_{12\,15} + E_{14\,17} + E_{16\,18} + E_{25\,26}$

$e_5 = \tilde{f}_5 = E_{5\,7} + E_{8\,10} + E_{9\,12} + E_{11\,14} + E_{13\,16} + E_{26\,27}$

$$e_6=\tilde{f}_6=E_{4\,6}+E_{5\,8}+E_{7\,10}+E_{19\,21}+E_{20\,22}+E_{23\,24}$$

$$h_1=E_{1\,1}-E_{2\,2}+E_{11\,11}-E_{13\,13}+E_{14\,14}-E_{16\,16}+E_{17\,17}-E_{18\,18}+E_{19\,19}$$
$$-E_{20\,20}+E_{21\,21}-E_{22\,22}$$

$$h_2=E_{2\,2}-E_{3\,3}+E_{9\,9}-E_{11\,11}+E_{12\,12}-E_{14\,14}+E_{15\,15}-E_{17\,17}+E_{20\,20}$$
$$+E_{22\,22}-E_{23\,23}-E_{24\,24}$$

$$h_3=E_{3\,3}-E_{4\,4}+E_{8\,8}-E_{9\,9}+E_{10\,10}-E_{12\,12}+E_{17\,17}+E_{18\,18}-E_{19\,19}$$
$$-E_{20\,20}+E_{24\,24}-E_{25\,25}$$

$$h_4=E_{4\,4}-E_{5\,5}+E_{6\,6}-E_{8\,8}+E_{12\,12}+E_{14\,14}-E_{15\,15}+E_{16\,16}-E_{17\,17}$$
$$-E_{18\,18}+E_{25\,25}-E_{26\,26}$$

$$h_5=E_{5\,5}-E_{7\,7}+E_{8\,8}+E_{9\,9}-E_{10\,10}+E_{11\,11}-E_{12\,12}+E_{13\,13}-E_{14\,14}$$
$$-E_{16\,16}+E_{26\,26}-E_{27\,27}$$

$$h_6=E_{4\,4}+E_{5\,5}-E_{6\,6}+E_{7\,7}-E_{8\,8}-E_{10\,10}+E_{19\,19}+E_{20\,20}-E_{21\,21}$$
$$-E_{22\,22}+E_{23\,23}-E_{24\,24}. \tag{42}$$

From (41) we have

$$h_0=-h_1-2h_2-3h_3-2h_4-h_5-2h_6$$
$$=-E_{1\,1}-E_{2\,2}-E_{3\,3}-E_{4\,4}-E_{5\,5}-E_{7\,7}+E_{21\,21}+E_{22\,22}$$
$$+E_{24\,24}+E_{25\,25}+E_{26\,26}+E_{27\,27} \tag{43}$$

$$k_0=q^{h_0/2}.$$

The embedding e_0 for $q\neq 1$ is the same as that for $q=1$:

$$e_0=\tilde{f}_0=E_{21\,1}+E_{22\,2}+E_{24\,3}+E_{25\,4}+E_{26\,5}+E_{27\,7}. \tag{44}$$

It is straightforward to check that (42), (43) and (44) satisfy the quantum algebraic relations (4) with $j=0,1,2,\ldots,6$. Through a tedious calculation we obtain the non-vanishing elements of $X(q)$ and $Y(q)$ satisfying the following relations:

$$X(q)_{N(m+r_0),Nm}=Y(q)_{N(m+r_0),Nm} \tag{45a}$$

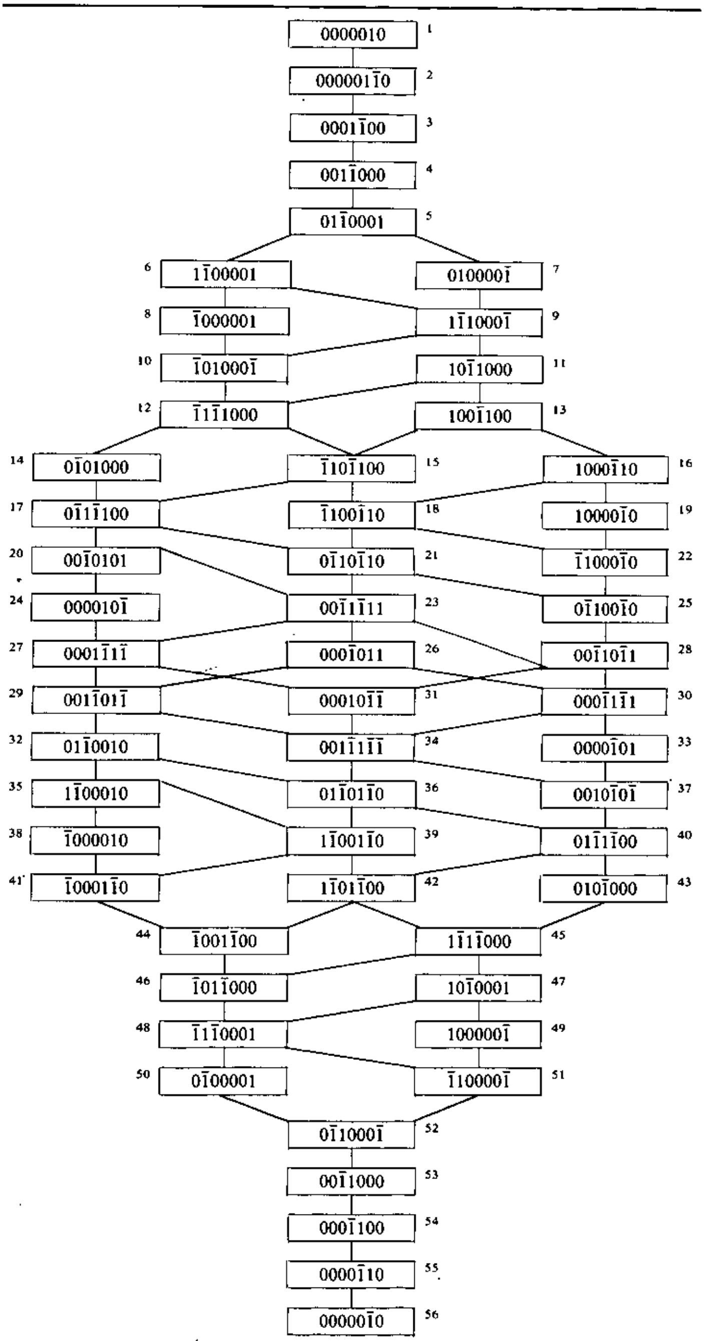
where $N=N_1$, N_2 and N_3, respectively, and

$$X(q)_{N'(m+r_0),Nm}=-q^{C_2(N')-C_2(N)}Y(q)_{N'(m+r_0),Nm} \tag{45b}$$

where the pair (N,N') or (N',N) denotes (N_1,N_2) and (N_2,N_3). Therefore, we obtain the same spectrum-dependent solution as that given in [11]:

$$\check{R}_q(x)=(1-xq^2)(1-xq^8)(C_q)_{N_1}(\tilde{C}_q)_{N_1}+(x-q^2)(1-xq^8)(C_q)_{N_2}(\tilde{C}_q)_{N_2} \tag{46}$$
$$+(x-q^2)(x-q^8)(C_q)_{N_3}(\tilde{C}_q)_{N_3}.$$

Table 3. Enumeration of 56 states in the minimal representation N_0 of E_7.



6. Solution for $q-E_7$

The Dynkin diagram of E_7 is shown in figure 3.

The decomposition of the direct product of the minimal representation $N_0=\lambda_6=(0000010)$ is multiplicity free:

$$(0000010)\otimes(0000010)=(0000020)\oplus(0000100)\oplus(1000000)\oplus(0000000) \qquad (47a)$$

or briefly

$$N_0\otimes N_0=N_1\oplus N_2\oplus N_3\oplus N_4 \qquad (47b)$$

where $N_1=2\lambda_6$, $N_2=\lambda_5$, $N_3=\lambda_1$ and $N_4=0$. The Casimir operator $C_2(N)$ is given as follows:

$$C_2(N_0)=57/4 \qquad C_2(N_1)=30 \qquad C_2(N_2)=28$$
$$C_2(N_3)=18 \qquad C_2(N_4)=0. \qquad (48)$$

The lowest negative root r_0 is

$$r_0=-\lambda_1=-2r_1-3r_2-4r_3-3r_4-2r_5-r_6-2r_7. \qquad (49)$$

The states in N_0 are enumerated as in table 3 so that the sum of the enumerations of m and $\bar{m}=-m$ is 57.

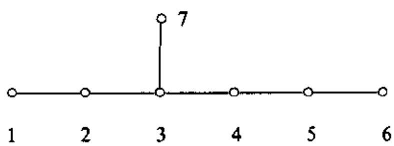


Figure 3. Dynkin diagram of E_7.

For the quantum E_7 universal enveloping algebra, the representation matrices of e_j, f_j and h_j in N_0 are given as follows:

$$e_1=\tilde{f}_1=E_{6\,8}+E_{9\,10}+E_{11\,12}+E_{13\,15}+E_{16\,18}+E_{19\,22}+E_{35\,38}+E_{39\,41}$$
$$+E_{42\,44}+E_{45\,46}+E_{47\,48}+E_{49\,51}$$

$$e_2=\tilde{f}_2=E_{5\,6}+E_{7\,9}+E_{12\,14}+E_{15\,17}+E_{18\,21}+E_{22\,25}+E_{32\,35}+E_{36\,39}$$
$$+E_{40\,42}+E_{43\,45}+E_{48\,50}+E_{51\,52}$$

$$e_3=\tilde{f}_3=E_{4\,5}+E_{9\,11}+E_{10\,12}+E_{17\,20}+E_{21\,23}+E_{25\,28}+E_{29\,32}+E_{34\,36}$$
$$+E_{37\,40}+E_{45\,47}+E_{46\,48}+E_{52\,53}$$

$$e_4=\tilde{f}_4=E_{3\,4}+E_{11\,13}+E_{12\,15}+E_{14\,17}+E_{23\,26}+E_{27\,29}+E_{28\,30}+E_{31\,34}$$
$$+E_{40\,43}+E_{42\,45}+E_{44\,46}+E_{53\,54}$$

$$e_5=\tilde{f}_5=E_{2\,3}+E_{13\,16}+E_{15\,18}+E_{17\,21}+E_{20\,23}+E_{24\,27}+E_{30\,33}+E_{34\,37}$$
$$+E_{36\,40}+E_{39\,42}+E_{41\,44}+E_{54\,55}$$

$$e_6=\tilde{f}_6=E_{1\,2}+E_{16\,19}+E_{18\,22}+E_{21\,25}+E_{23\,28}+E_{26\,30}+E_{27\,31}+E_{29\,34}$$
$$+E_{32\,36}+E_{35\,39}+E_{38\,41}+E_{55\,56}$$

$$e_7=\tilde{f}_7=E_{5\,7}+E_{6\,9}+E_{8\,10}+E_{20\,24}+E_{23\,27}+E_{26\,29}+E_{28\,31}+E_{30\,34}$$
$$+E_{33\,37}+E_{47\,49}+E_{48\,51}+E_{50\,52}$$
$$h_1=E_{6\,6}-E_{8\,8}+E_{9\,9}-E_{10\,10}+E_{11\,11}-E_{12\,12}+E_{13\,13}-E_{15\,15}+E_{16\,16}$$
$$-E_{18\,18}+E_{19\,19}-E_{22\,22}+E_{35\,35}-E_{38\,38}+E_{39\,39}-E_{41\,41}+E_{42\,42}$$
$$-E_{44\,44}+E_{45\,45}-E_{46\,46}+E_{47\,47}-E_{48\,48}+E_{49\,49}-E_{51\,51}$$
$$h_2=E_{5\,5}-E_{6\,6}+E_{7\,7}-E_{9\,9}+E_{12\,12}-E_{14\,14}+E_{15\,15}-E_{17\,17}+E_{18\,18}-E_{21\,21}$$
$$+E_{22\,22}-E_{25\,25}+E_{32\,32}-E_{35\,35}+E_{36\,36}-E_{39\,39}+E_{40\,40}-E_{42\,42}$$
$$+E_{43\,43}-E_{45\,45}+E_{48\,48}-E_{50\,50}+E_{51\,51}-E_{52\,52}$$
$$h_3=E_{4\,4}-E_{5\,5}+E_{9\,9}+E_{10\,10}-E_{11\,11}-E_{12\,12}+E_{17\,17}-E_{20\,20}+E_{21\,21}-E_{23\,23}$$
$$+E_{25\,25}-E_{28\,28}+E_{29\,29}-E_{32\,32}+E_{34\,34}-E_{36\,36}+E_{37\,37}-E_{40\,40}+E_{45\,45}$$
$$+E_{46\,46}-E_{47\,47}-E_{48\,48}+E_{52\,52}-E_{53\,53}$$
$$h_4=E_{3\,3}-E_{4\,4}+E_{11\,11}+E_{12\,12}-E_{13\,13}+E_{14\,14}-E_{15\,15}-E_{17\,17}+E_{23\,23}$$
$$-E_{26\,26}+E_{27\,27}+E_{28\,28}-E_{29\,29}-E_{30\,30}+E_{31\,31}-E_{34\,34}+E_{40\,40}+E_{42\,42}$$
$$-E_{43\,43}+E_{44\,44}-E_{45\,45}-E_{46\,46}+E_{53\,53}-E_{54\,54}$$
$$h_5=E_{2\,2}-E_{3\,3}+E_{13\,13}+E_{15\,15}-E_{16\,16}+E_{17\,17}-E_{18\,18}+E_{20\,20}-E_{21\,21}$$
$$-E_{23\,23}+E_{24\,24}-E_{27\,27}+E_{30\,30}-E_{33\,33}+E_{34\,34}+E_{36\,36}-E_{37\,37}+E_{39\,39}$$
$$-E_{40\,40}+E_{41\,41}-E_{42\,42}-E_{44\,44}+E_{54\,54}-E_{55\,55}$$
$$h_6=E_{1\,1}-E_{2\,2}+E_{16\,16}+E_{18\,18}-E_{19\,19}+E_{21\,21}-E_{22\,22}+E_{23\,23}-E_{25\,25}$$
$$+E_{26\,26}+E_{27\,27}-E_{28\,28}+E_{29\,29}-E_{30\,30}-E_{31\,31}+E_{32\,32}-E_{34\,34}+E_{35\,35}$$
$$-E_{36\,36}+E_{38\,38}-E_{39\,39}-E_{41\,41}+E_{55\,55}-E_{56\,56}$$
$$h_7=E_{5\,5}+E_{6\,6}-E_{7\,7}+E_{8\,8}-E_{9\,9}-E_{10\,10}+E_{20\,20}+E_{23\,23}-E_{24\,24}+E_{26\,26}$$
$$-E_{27\,27}+E_{28\,28}-E_{29\,29}+E_{30\,30}-E_{31\,31}+E_{33\,33}-E_{34\,34}-E_{37\,37}+E_{47\,47}$$
$$+E_{48\,48}-E_{49\,49}+E_{50\,50}-E_{51\,51}-E_{52\,52}. \tag{50}$$

From (49) we have

$$h_0=-2h_1-3h_2-4h_3-3h_4-2h_5-h_6-2h_7$$
$$=-E_{1\,1}-E_{2\,2}-E_{3\,3}-E_{4\,4}-E_{5\,5}-E_{6\,6}-E_{7\,7}-E_{9\,9}-E_{11\,11}-E_{13\,13}$$
$$-E_{16\,16}-E_{19\,19}+E_{38\,38}+E_{41\,41}+E_{44\,44}+E_{46\,46}+E_{48\,48}+E_{50\,50}$$
$$+E_{51\,51}+E_{52\,52}+E_{53\,53}+E_{54\,54}+E_{55\,55}+E_{56\,56}. \tag{51}$$

The embedding e_0 for $q\neq 1$ is the same as that for $q=1$:

$$e_0=\tilde{f}_0=E_{38\,1}+E_{41\,2}+E_{44\,3}+E_{46\,4}+E_{48\,5}+E_{50\,6}+E_{51\,7}+E_{52\,9}+E_{53\,11}$$
$$+E_{54\,13}+E_{55\,16}+E_{56\,19}. \tag{52}$$

Equations (50), (51) and (52) satisfy the quantum algebraic relations (4) with $j = 0, 1, 2, \ldots, 7$. The non-vanishing elements of $X(q)$ and $Y(q)$ satisfy the following relations:

$$X(q)_{N(m+r_0),Nm} = Y(q)_{N(m+r_0),Nm} \tag{53a}$$

where $N = N_1$, N_2, N_3 and N_4, respectively, and

$$X(q)_{N'(m+r_0),Nm} = -q^{C_2(N')-C_2(N)}\, Y(q)_{N'(m+r_0),Nm} \tag{53b}$$

where the pair (N, N') or (N', N) denotes (N_1, N_2), (N_2, N_3) and (N_3, N_4). Therefore, we have proved that the spectrum-dependent solution given in [11],

$$\begin{aligned}\check{R}_q(x) = {} & (1-xq^2)(1-xq^{10})(1-xq^{18})(C_q)_{N_1}(\tilde{C}_q)_{N_1} \\ & + (x-q^2)(1-xq^{10})(1-xq^{18})(C_q)_{N_2}(\tilde{C}_q)_{N_2} \\ & + (x-q^2)(x-q^{10})(1-xq^{18})(C_q)_{N_3}(\tilde{C}_q)_{N_3} \\ & + (x-q^2)(x-q^{10})(x-q^{18})(C_q)_{N_4}(\tilde{C}_q)_{N_4}\end{aligned} \tag{54}$$

does satisfy the Yang–Baxter equation.

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References

[1] Faddeav L 1984 Integrable models in (1 + 1)-dimensional quantum field theory *Les Houches Lecture* (Amsterdam: Elsevier)
[2] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[3] Date E, Jimbo M, Kuniba A, Miwa T and Okado M 1989 *Lett. Math. Phys.* **17** 69
[4] Akutsu Y and Wadati M 1987 *J. Phys. Soc. Japan* **56** 3039
[5] Moore G and Seiberg N 1989 Lectures on RCFT, RU-89-32
[6] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[7] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[8] Kuniba A 1990 *J. Phys. A: Math. Gen.* **23** 1349
[9] Koh I G and Ma Z Q 1990 *Phys. Lett.* **234B** 480
[10] Kim J D, Koh I G and Ma Z Q 1990 In preparation
[11] Ma Zhong-Qi 1990 The spectrum-dependent solutions to Yang–Baxter equation for quantum E_6 and E_7 *J. Phys. A: Math. Gen.* **23**
[12] Chung H J and Koh I G 1990 Solutions to the quantum Yang–Baxter equation for the exceptional Lie algebras with a spectral parameter *Preprint*
[13] Zhang R B, Gould M D and Bracken A J 1989 From representations of the braid group to solutions of the Yang–Baxter equation *Preprint*
[14] Ogievetsky E and Wiegmann P 1986 *Phys. Lett.* **168B** 360
[15] Ogievetsky E, Reshetikhin N and Wiegmann P 1987 *Nucl. Phys.* B **280** 45
[16] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[17] Hou Bo-Yu, Hou Bo-Yuan, Ma Zhong-Qi and Yin Yu-Dong 1990 Solutions of Yang–Baxter equation in the vertex model and the face model for octet representation *Preprint BIHEP-TH-90-4*
[18] Hou Bo-Yu, Hou Bo-yuan, Ma Zhong-Qi and Yin Yu-Dong 1990 Rational solution to Yang–Baxter equation in the octet representation *Preprint BIHEP-TH-90-12*